Analytic Results on Dark Matter Velocity Distributions

Jason Kumar
University of Hawaii

collaborators

- Katharena Christy
- Taylor Herbert
- Louis E. Strigari

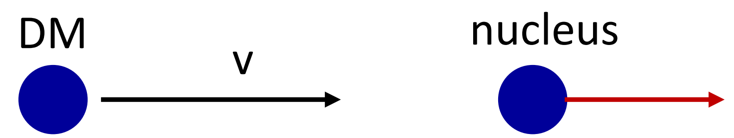
- PRD**109** 063016 (2024) [2309.01979]
- 2xxx.xxxxx

dark matter velocity distribution

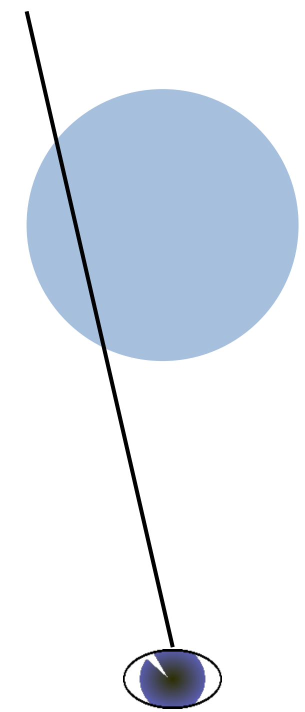
- why do we care about dark matter velocity distributions?

- direct detection → energy deposited must exceed threshold

- indirect detection → annihilation cross section can depend on v
  - velocity distribution tells us where and how much annihilates
  - affects photon angular distribution
  - relevant for GC, nearby subhalos



$$E_R^{max} = \frac{2\mu^2 v^2}{m_A}$$



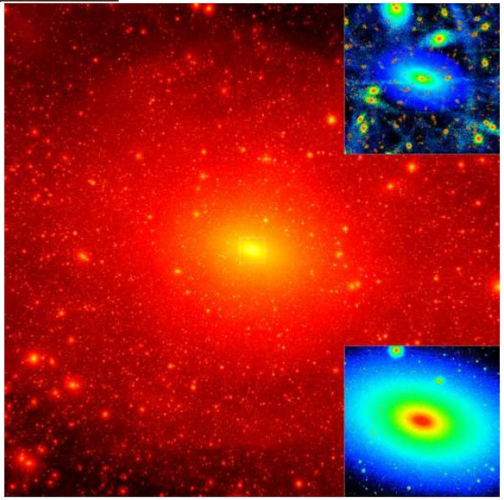
methods for studying velocity dist.

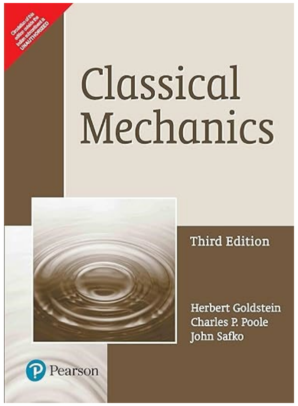
- observation
  - use motions of stars to trace dark matter kinematics
- numerical simulation
  - simulate a large number of particles interacting via gravity
- analytic methods
  - make some approximations, and then find general connection between density and velocity

Do they agree?

GAIA

Via Lactea 2
0805.1244





standard scenario

- often assumed to be a Maxwell-Boltzmann distribution
  - typical assumption for direct detection experiments
- motivated by isothermal models
  - flat rotation curve, fixed velocity dispersion
  - $\rho(r) \propto r^{-2}$
  - may not be a good description
  - NFW often seen in simulation
  - $\rho(r) = \rho_s [r/r_s]^{-1} [1+(r/r_s)]^{-2}$
- large N simulations show some consistency with MB
  - better for simulations with baryons (Piccirillo, et al., 2203.08853)
  - **see Nassim's talk**
- evidence for DM streams in MW, simulations
  - likely there are small-scale deviations (Necib, et al., 1807.02519)
- velocity dispersion has to decrease as radius becomes large
  - halo has to truncate

there are questions to be answered ….

outline

- how can we learn about velocity distributions using classical mechanics?

- do these analytic results match with numerical simulations?

assumptions/approximations

i. assume matter distribution is spherically symmetric

ii. assume matter distribution is static (time-averaged distribution is a good approximation today)

iii. assume dark matter particles subject to a central force which depends only on radial position
   - if forces are gravitational only, then first two assumptions imply the third

iv. assume dark matter velocity distribution is isotropic (*optional*)

none of the assumptions are true

- generally not spherically symmetric
  - simulations generally find some level of triaxiality, even in DM-only case
  - baryon distribution typically not spherically symmetric (disk, etc.)

- not static → merger history is important
  - simulations show noticeable features due to late mergers
  - see these effects also in observation of Milky Way with GAIA

- generally anisotropic
  - dependence on velocity direction, not just speed
  - also related to merger history

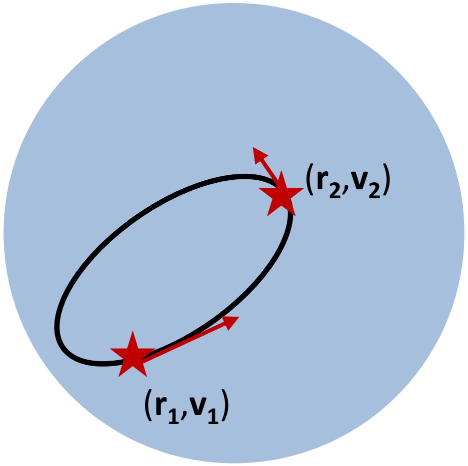
- but if none of the assumptions are true, then why make them?

goal of assumptions

- want to determine the consequences of each approximation, even if not exactly true
- can help understand how deviations from analytic predictions can be traced back to deviations from underlying assumptions
- how important are the deviations from assumptions to coarse-grained predictions?
- starting point from assumptions i – iii …
- … this is essentially a central potential problem

problem in classical mechanics

- vel. dist. is a phase space density
- Liouville’s theorem …
  - under canonical transformation, phase space volume invariant
  - time translation is a canonical transformation
- … so phase space density is invariant under time translation



- average velocity distribution is constant on classical path
- lets us solve for the velocity distribution, …

… and a nice example of how advanced topics in classical mechanics are relevant to fundamental research in astrophysics

vel. dist. and integrals of motion

- 6 integrals of motion fix path
- for central potential
  - $\psi_{1,2,3}$ specify orientation of path
  - E, L
  - t_0
- static and spherically symmetric: reduces to three variables
- Liouville's theorem: indep. of t_0
- isotropy: indep. of L

- $f(r_1, r_2, r_3, v_1, v_2, v_3, t)$
- $f(\psi_1, \psi_2, \psi_3, E, L, t_0, t)$

- $f(r, v_r, v_\perp)$
- $f(E, L, t_0)$

- $f(r, v_r, v_\perp)_{constrained}$
- $f(E, L)$

- $f(r, v)_{constrained}$
- $f(E)$

$f(E) = f(E(r,v))$

Eddington inversion

- much easier if isotropic
  → f(E,L) = f(E)
- can perform L integral

$$\rho(r) = \int_0^{v_{esc}} \int dv_r \, dv_\perp^2 \, f(r, v_r, v_\perp)$$

$$= 2\sqrt{2}\pi \int_0^{\sqrt{2}r\sqrt{\Phi(\infty)-\Phi(r)}} dL \int_{\frac{L^2}{2r^2}+\Phi(r)}^{\Phi(\infty)} dE \frac{L^2}{r^2} \frac{f(E,L)}{\sqrt{E - (L^2/2r^2) - \Phi(r)}}$$

- ρ and f then related by an Abel integral transform
- can do an inverse transform to get f from ρ
  - Eddington inversion (MNRAS, 76, 572, 1916)

$$E = \frac{1}{2}\left(v_r^2 + v_\perp^2\right) + \Phi(r) = \text{energy / mass}$$

$$\Phi(r) = \text{gravitational potential}$$

- given profile, get velocity-distribution numerically

$$\rho(r) = 4\sqrt{2}\pi \int_{\Phi(r)}^{\Phi(\infty)} dE \, \sqrt{E - \Phi(r)} \, f(E)$$

$$f(E) = \frac{1}{\sqrt{8}\pi^2} \int_E^{\Phi(\infty)} \frac{d^2\rho}{d\Phi^2} \frac{d\Phi}{\sqrt{\Phi - E}}$$

scaling

- focus on cusp, take $\Phi(\infty) \gg \Phi(r)$
  - DM density largest (take $\Phi(0)=0$)
- $\rho(r)$ derived from integrating $f(E)$ over E accessible at r
- $f(E)$ derived from integrating $\rho(r)$ over r inaccessible at E

$$\rho(r) = 2\sqrt{2}\pi \int_0^\infty dL \int_{\frac{L^2}{2r^2}+\Phi(r)}^{\infty} dE \frac{L^2}{r^2} \frac{f(E,L)}{\sqrt{E-\left(L^2/2r^2\right)-\Phi(r)}}$$

- so inversion formula is exactly correct and unique, …
- … but only if f is a function of E alone everywhere

$$\rho(r) = 4\sqrt{2}\pi \int_{\Phi(r)}^{\infty} dE \sqrt{E-\Phi(r)}\, f(E)$$

$$f(E) = \frac{1}{\sqrt{8}\pi^2} \int_E^\infty \frac{d^2\rho}{d\Phi^2} \frac{d\Phi}{\sqrt{\Phi - E}}$$

- we can find an analytic approx ….

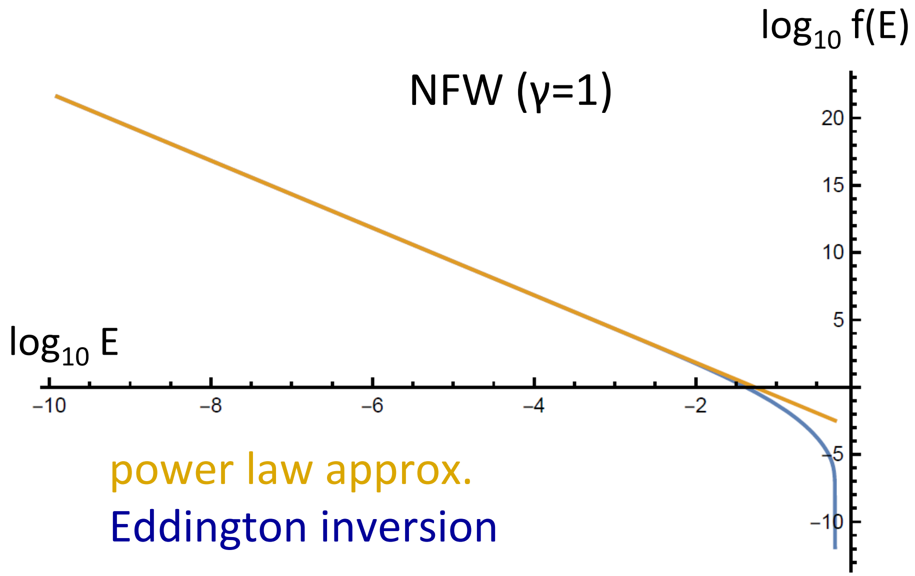
$\Phi(r)$ sets the scale of E, and vice versa

near the cusp

- much simpler in the cusp
- $\rho(r) \propto r^{-\gamma}$
- $\Phi(r) \propto r^{2-\gamma}$ (DM-only, $\Phi(0)=0$)
- $f(E) \propto E^{(\gamma-6)/[2(2-\gamma)]}$

- power-law matches the exact result at small E (small r)

- fails at large E, but that's where density is small
  - high-speed particles can explore outside the cusp
  - need to know the details



dependence on scale radius r_s and scale density ρ_s determined by dimensional analysis

analytic methods give functional dependence of f(E) on halo parameters

power laws and Boltzmann

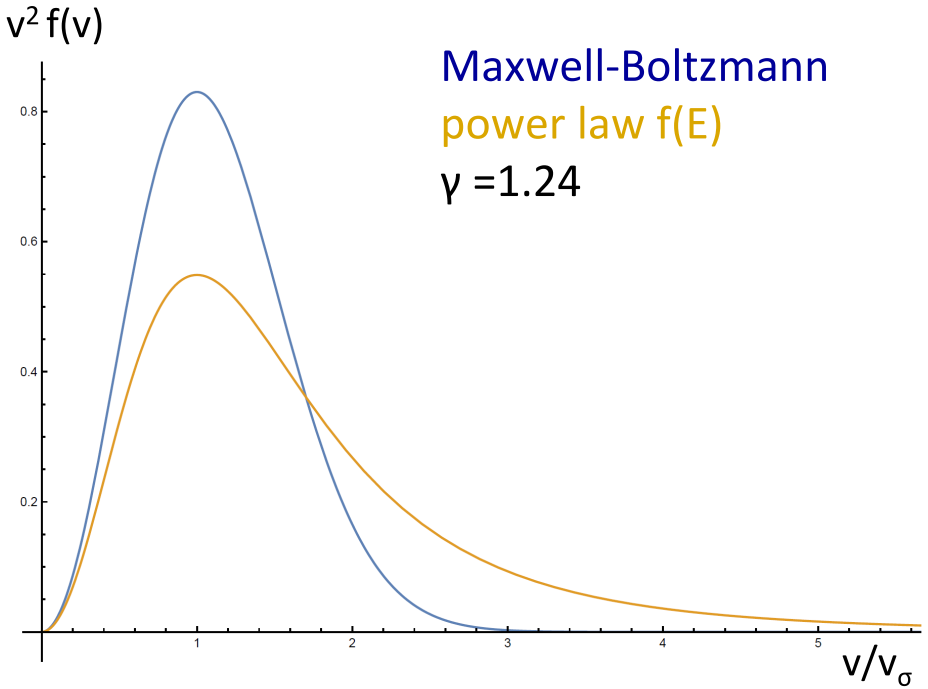
- analytic results → if ρ and Φ are power law, so is f(E)
- standard approach → Maxwell-Boltzmann (decent fit to N-body)

$$f_{MB}(v) \propto \exp\left[-v^2 / 2v_\sigma^2\right]$$

v_σ depends on r

- similar, but high-v tail differs
- important for p-/d-wave annih., scattering of low-mass DM

- can we compare analytic results to N-body simulation results?

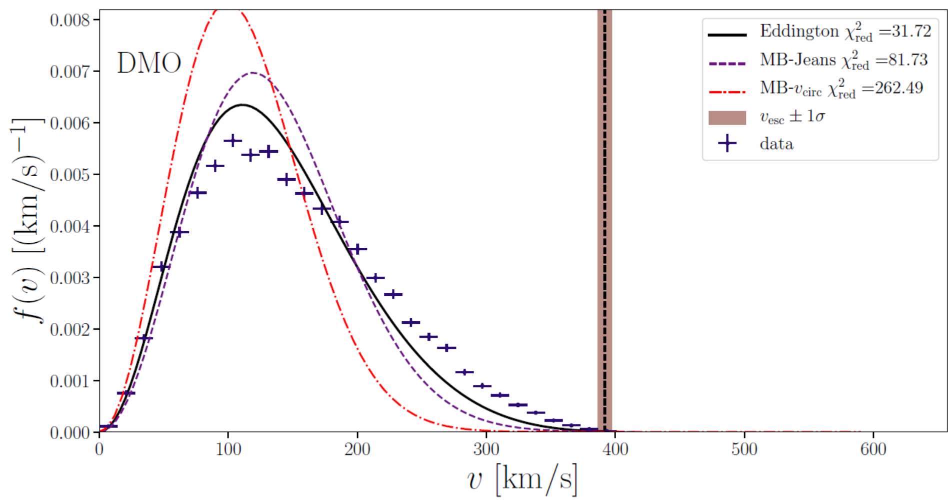


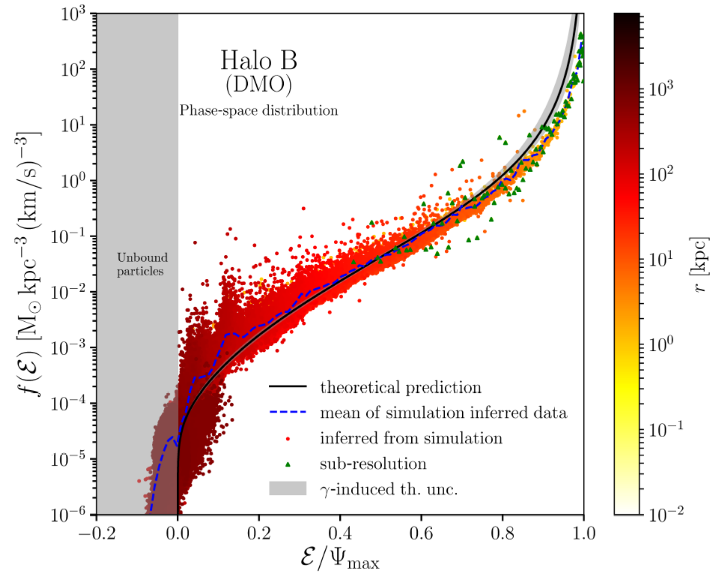
previous results

- previous study of 3 MW-sized halos sims (w/ or w/o baryons) (Lacroix, et al. 2005.03955)
- general preference for Eddington vs. Maxwell-Boltzmann
- but quantitatively, not great
  - χ^2/dof ~ $\mathcal{O}(10)$
  - f(E) varies with r by ~ $\mathcal{O}(100)$
- did not focus on innermost part of the cusp
  - merger effects less important
- didn't compare fit in different radial bins to f(E)

2005.03955

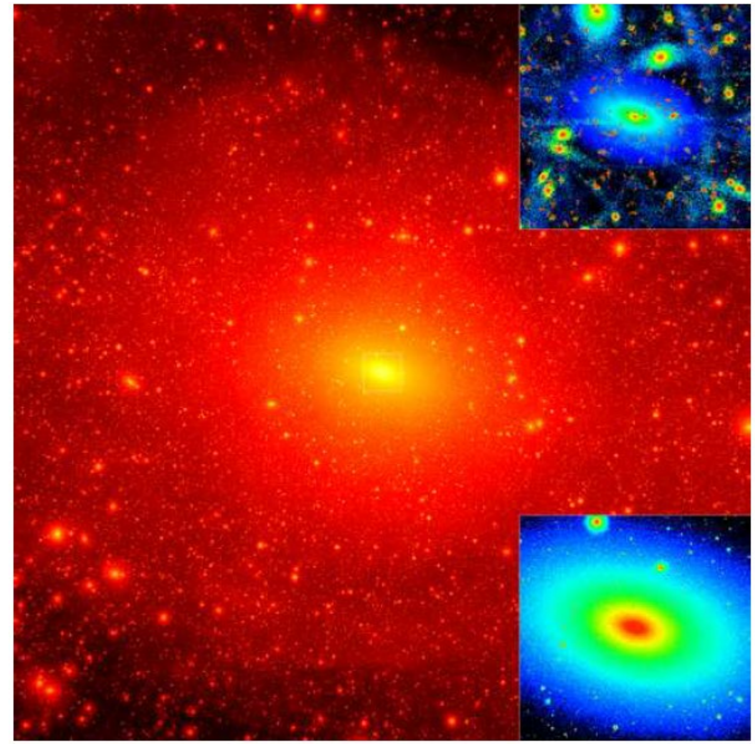
2-4 kpc





VL-2

- let's compare to Via Lactea 2
  - MW-sized halo simulation, DM-only, 10^9 particles (0805.1244)
- best fit →
  - gNFW, $\gamma=1.24$
  - r_s = 28.1 kpc
  - $\rho_s = 0.0035\ M_\odot / pc^3$
  - convergence radius = 0.38 kpc
- 10^5 particles publicly available
  - reasonably spherical
  - well fit to gNFW out to $\sim 24\ r_s$



Via Lactea 2
0805.1244

$$\rho(r) = \frac{\rho_s}{(r/r_s)^{\gamma}\left[1+(r/r_s)\right]^{3-\gamma}}$$

testing assumptions

- can test equilibrium and isotropy

- if in equilibrium, should have q=1, from virial theorem
- find q=1.09 for $r < 24\, r_s$

- β is spherical anisotropy parameter
  - β=0 for isotropic vel. dist.
- note, β is a spherically-averaged measure of isotropy
  - for VL-2, there is anisotropy, but averages out

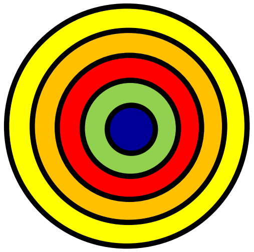
Φ(r) from data differs from gNFW by < 1%

$$q \equiv \frac{2\sum E_{kin}}{-\sum \vec{F}\cdot\vec{r}}$$

$$\beta \equiv 1 - \frac{\langle v_\perp^2 \rangle}{2\langle v_r^2 \rangle}$$

comparing to numerical simulations

- focus on range $r/r_s < 0.5$
  - largest density
  - merger effects less pronounced
  - closer to isotropic, so f(E)
- divide into 5 radial bins
- expand density in spherical harmonics ($\ell = 0\ ,\ 1$)
- compute β and uncertainty
  - just propagate error linearly
- at a coarse-grained level, spherical symmetry and isotropy seem not unreasonable
- how reasonable?

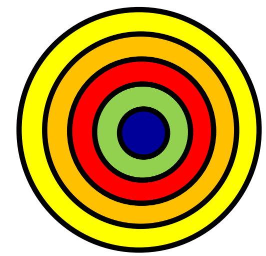


region	a_{00}	a_{10}	Re a_{11}	Im a_{11}	β
$0 < \tilde{r} < 0.1$ (A)	0.28	0.0028	0.0095	0.0000198	0.04 ± 0.12
$0.1 < \tilde{r} < 0.2$ (B)	0.28	0.01	0.01	0.0024	0.03 ± 0.08
$0.2 < \tilde{r} < 0.3$ (C)	0.28	0.01	0.007	0.027	0.14 ± 0.05
$0.3 < \tilde{r} < 0.4$ (D)	0.28	−0.005	0.008	0.02	0.16 ± 0.05
$0.4 < \tilde{r} < 0.5$ (E)	0.28	0.001	−0.015	0.022	0.13 ± 0.04

$\tilde{r} = r/r_s$

getting the velocity distribution

- divide each radial bin into two subregions
  - bin in energy
- compute f(E) in all 10 radial regions
  - compare to each other, and to Eddington result (2110.09653)
- distinct regions of phase space
  - not related by rotation
  - related by integrals of motion



$$\rho(r) = 4\sqrt{2}\pi \int_{\Phi(r)}^{\Phi(\infty)} dE \, \sqrt{E - \Phi(r)} \, f(E)$$

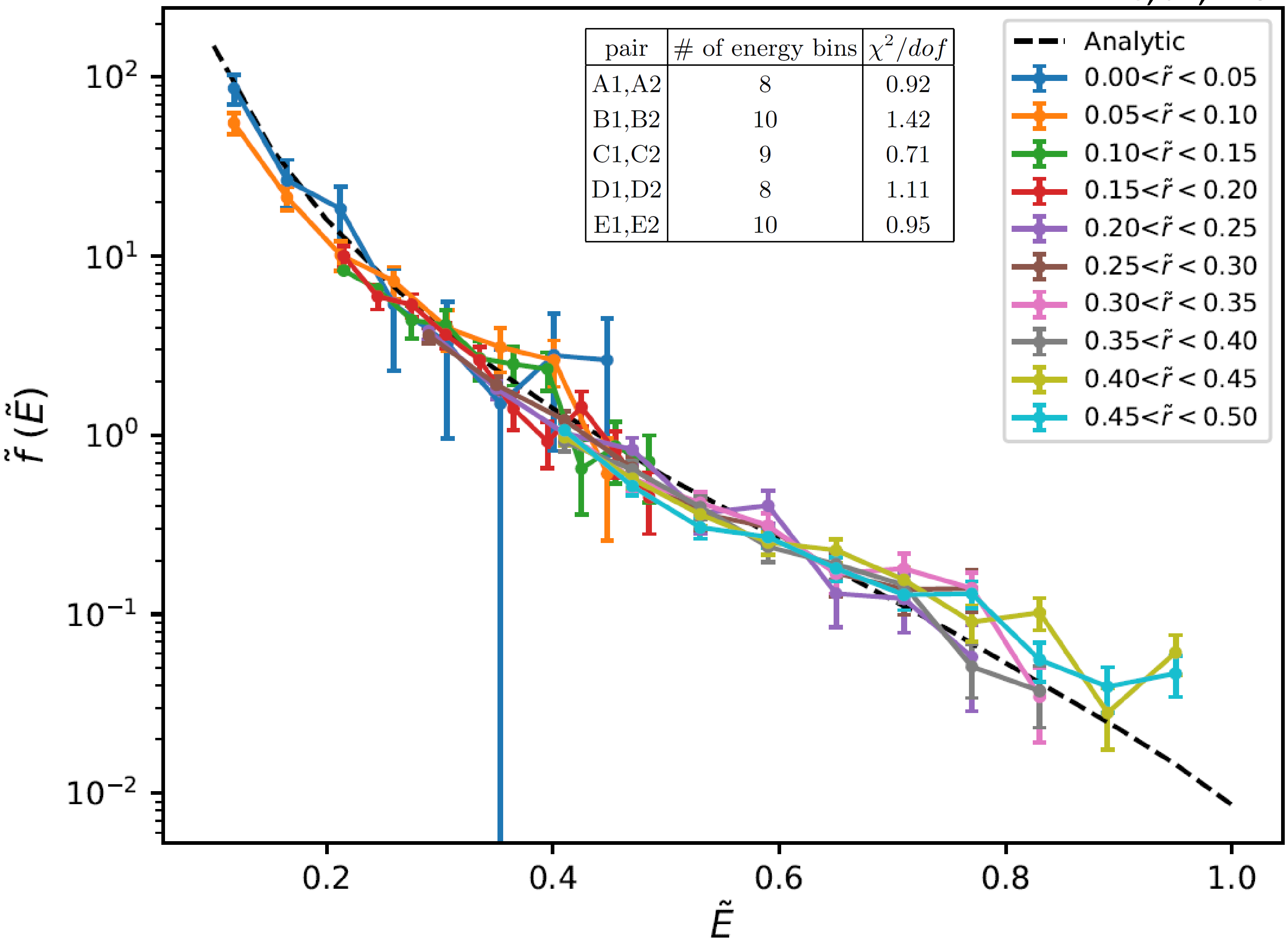
$$f(E) = \frac{N(E,r)}{4\sqrt{2}\pi\sqrt{E_{avg} - \Phi_{avg}(r)} \, \Delta E \, \Delta V}$$

f(E) should be same for every radial bin

range of allowed E (from Φ(r) to Φ(∞)) depends on r

result

KC, JK, LES 2309.01979



pair	# of energy bins	χ^2/dof
A1,A2	8	0.92
B1,B2	10	1.42
C1,C2	9	0.71
D1,D2	8	1.11
E1,E2	10	0.95

consistent with Eddington?

- note that Eddington inversion is slightly, but systematically, larger than data at intermediate E, systematically smaller at larger E
- f(E) from different radial bins are more consistent with each other than with Eddington result

- if f was a function of E everywhere, it would have to be the Eddington inversion result …
- … but if not, then Eddington inversion need not hold exactly

- maybe the small difference is related to this? Or just binning error?
- would need to understand dependence on L, and greater precision
- which means we'd need more than 10^5 particles

Auriga

- Auriga Project has publicly released complete data from 40 N-body simulations of MW-sized galaxies (2401.08750)
- includes baryon and DM-only runs
- possible to study dependence of f on E and L, with much greater precision
- study the effect of baryons on L dependence
- work in progress with Taylor Herbert (UH undergrad) …
  - … who is applying to grad schools in the fall!

conclusion

- dark matter velocity distributions can have an important impact on direct and indirect detection strategies
- can gain a lot of insight from analytic arguments
- can help with understanding low-mass DM, angular distribution of GC excess, etc.

- useful complement to the approach of using numerical simulations

Mahalo!

Backup Slides

velocity-dependent DM annihilation

- usual assumption is annihilation from s-wave state ($\sigma_A v$ = const.)
- for many models, it scales as v^n

1) p-wave annihilation ($n=2$)
   - initial state must be $L=1$
   - $\sigma_A v \propto v^2$
2) Sommerfeld-enhanced annihilation ($n=-1$)
   - there is a long-range attractive force between DM particles
   - $\sigma_A v \propto v^{-1}$ (Coulomb limit)

- say $\bar{X}X$ annihilate through intermediate $J^P = 0^+$ state
- $P = (-1)^{L+1}$ if X is a fermion
- need L = odd if parity conserved

- could also have annihilation from a d-wave state ($n=4$)

- how does non-trivial velocity dependence affect angular distribution?

number crunching

divide each radial region into two subregions
each region has same energy binning

$\sim$ 3600 particles with $r < 0.5\ r_s$

pair	# of energy bins	χ^2/dof
A1,A2	8	0.92
B1,B2	10	1.42
C1,C2	9	0.71
D1,D2	8	1.11
E1,E2	10	0.95

$$\delta\beta = \frac{\langle v_\perp^2 \rangle}{2\langle v_r^2 \rangle} \left[\frac{\langle v_r^4 \rangle - \langle v_r^2 \rangle^2}{N \langle v_r^2 \rangle^2} + \frac{\langle v_\perp^4 \rangle - \langle v_\perp^2 \rangle^2}{N \langle v_\perp^2 \rangle^2} \right]^{1/2}$$

gravitational potential in the cusp

$$\rho(r) \approx \rho_s \left(\frac{r}{r_s}\right)^{-\gamma}$$

$$\begin{aligned}\Phi(r) &= \int_0^r dx \frac{G_N}{x^2} \int_0^x dy \ \left(4\pi y^2\right) \ \rho(y) \\ &= 4\pi G_N \rho_s \int_0^r dx \frac{1}{x^2} \int_0^x dy \ y^2 \left(\frac{y}{r_s}\right)^{-\gamma} \\ &= \frac{4\pi G_N \rho_s r_s^3}{3-\gamma} \int_0^r dx \frac{1}{x^2} \left(\frac{x}{r_s}\right)^{3-\gamma} \\ &= \frac{4\pi G_N \rho_s r_s^2}{(3-\gamma)(2-\gamma)} \left(\frac{r}{r_s}\right)^{2-\gamma}\end{aligned}$$

$$\Phi(0) = 0$$

dark matter halos

Primack 1505.02821

- dark matter halos seed structure formation
- overdensities form small halos ….
- … seed formation of larger halos, host galaxies (including MW)

- care about halo density
- large N numerical simulations show formation of cuspy halos
  - though self-interaction can yield cores (not this talk)
- standard fit, generalized Navarro-Frenk-White profile (gNFW)

$$\rho(r) = \frac{\rho_s}{(r/r_s)^{\gamma}\left[1+(r/r_s)\right]^{3-\gamma}}$$

r_s = scale radius, ρ_s = scale density
γ = inner slope

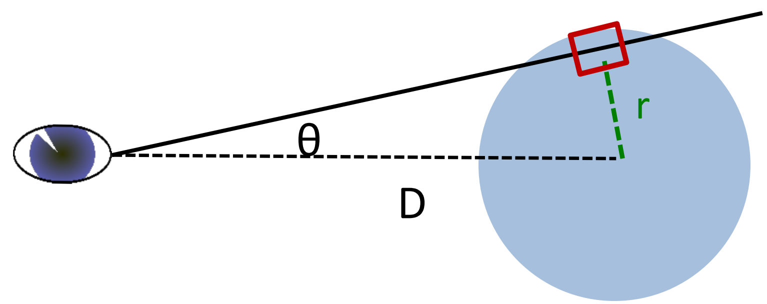
angular distribution

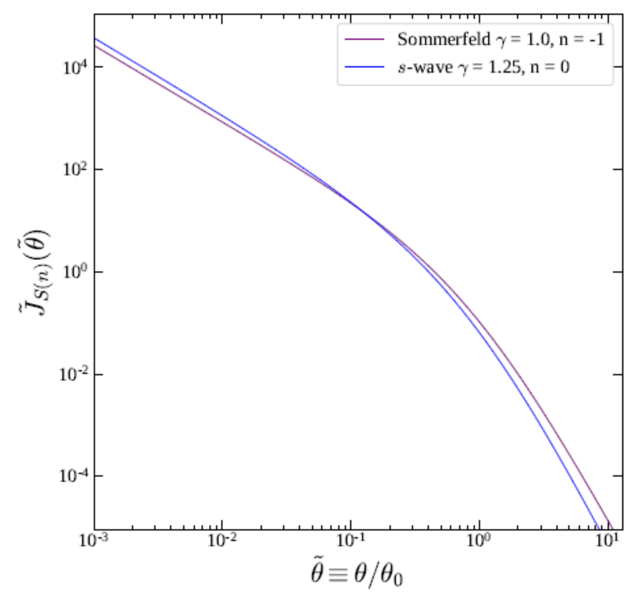
- annih. rate depends on f(r,v) along line of sight
- trade v for E in integral
- power-law dep. on θ at small θ
- but there's a degeneracy between n and γ
  - broken at larger angle
- for a big/near enough halo (GC, nearby dSph), can potentially measure angular distribution

- degeneracy good because high-v tail doesn't contribute for $n \le 0$

$$\frac{\text{rate}}{\text{vol}} \propto \int d^3v_1 d^3v_2\ f(\vec{r},\vec{v}_1)\ f(\vec{r},\vec{v}_2)\ \sigma_A v_{rel}$$

$$\sigma_A v \propto v^n$$



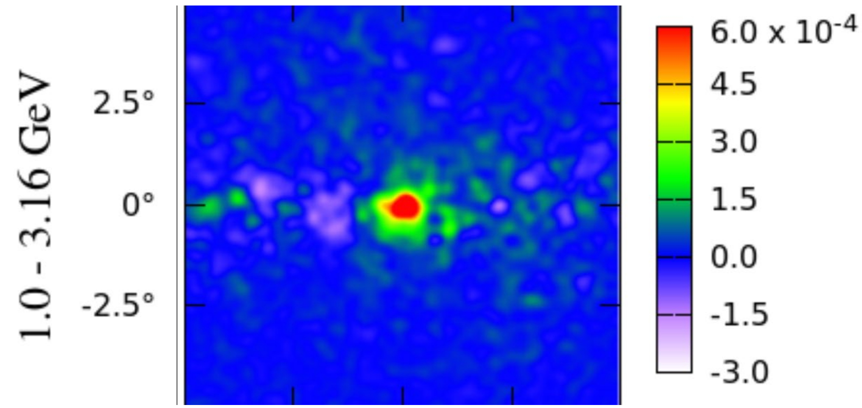


BB, JK, VL, JR 2110.09653
PRD**106** 023025 (2022)

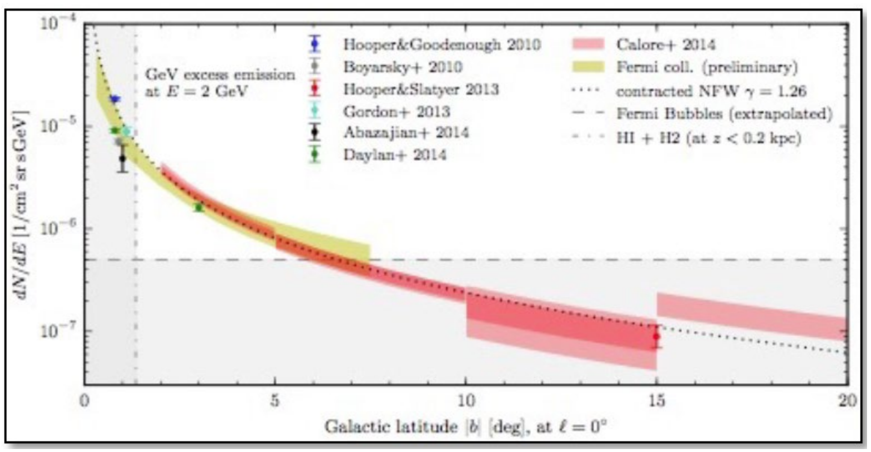
Galactic Center

- excess of photons (GeV range) seen from Galactic Center (Goodenough, Hooper 0910.2998, 1010.2752)
  - Fermi-LAT
- could be dark matter annihilation
- or could be millisecond pulsars
- lots of work studying this question from several angles

- I'll focus on DM hypothesis
- angular distribution decent fit to
  - s-wave annihilation (no v-dep.)
  - gNFW w/ γ=1.2



Hooper, PPC 2022

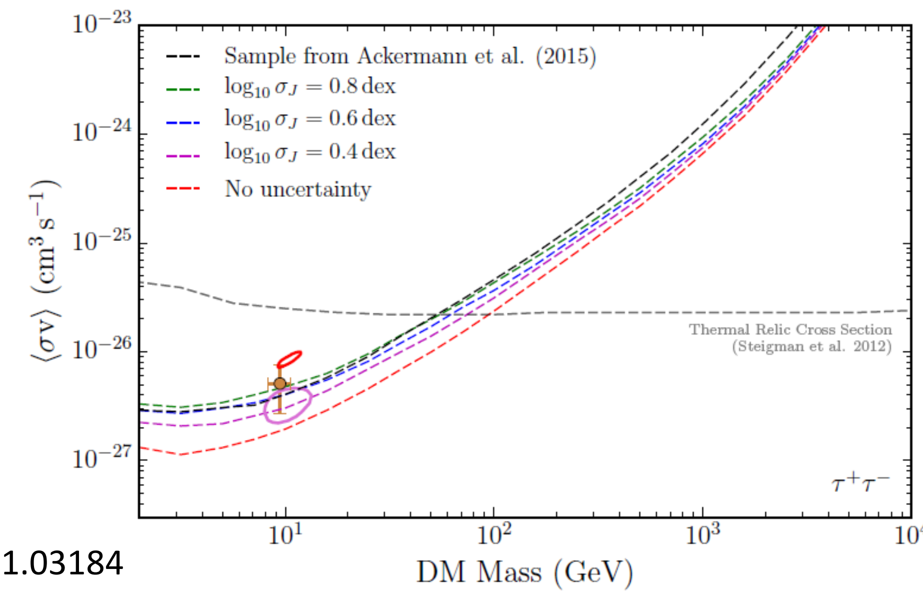


GC excess and p-wave

- dwarf spheroidal galaxy searches also constrain these models
- dSphs believed to be DM dominated → less background
- systematic uncertainties significant, but dSphs are starting to constrain GCE models
- but speeds are slower in dSphs than GC
- p-wave models (rate ∝ v^2) could weaken dSph constraints
- what happens to angular dist.?

Sculptor, JWST

Fermi

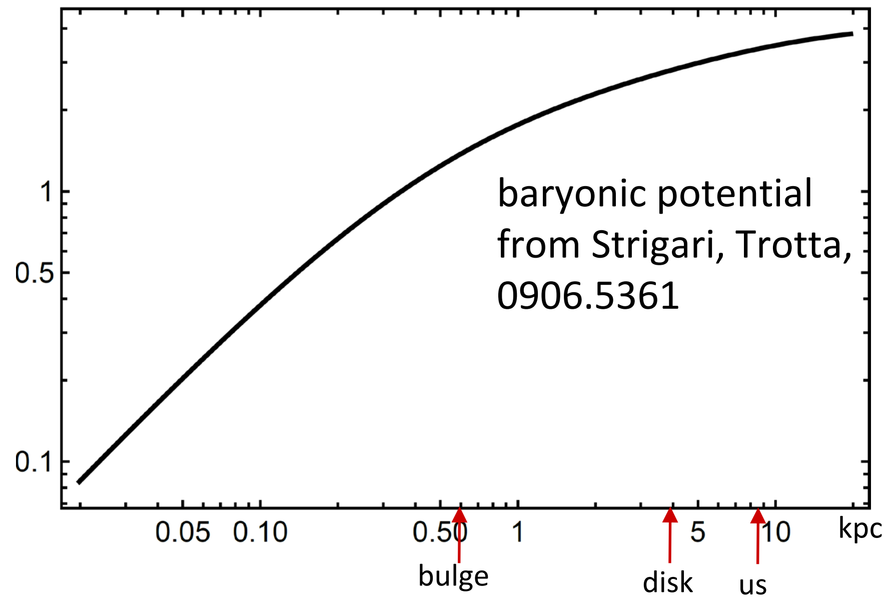


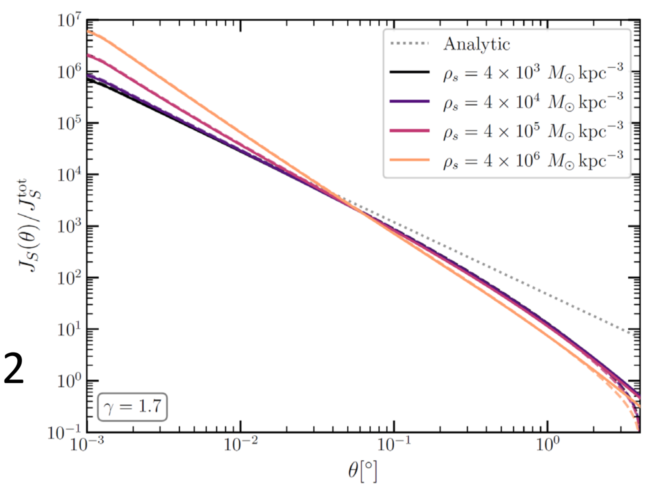
Fermi Collaboration, 1611.03184

does what happens in the bulge stay in the bulge?

- well inside galactic bulge, potential dominated by baryons
- same analytic arguments to predict angular distribution
- p-wave annih. dominated by high-v particles which see edge of bulge
- so even if ρ is power law to r_s …
- … potential is not, so velocity distribution not power law

- hard to get the angular distribution to match





KK, JK, JR 2208.14002
JCAP11 (2022) 030

Abel integral transform and inverse

$$F(y) = \int_y^{z_0} \frac{dx}{\sqrt{x-y}}\, G(x)$$

$$G(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^{z_0} \frac{dy}{\sqrt{y-x}}\, F(y)$$

$x, y < z_0$

$$G(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^{z_0} \frac{dy}{\sqrt{y-x}} \int_y^{z_0} \frac{ds}{\sqrt{s-y}}\, G(s)$$

$$= -\frac{1}{\pi}\frac{d}{dx}\int_x^{z_0} ds\, G(s) \int_x^{s} \frac{dy}{\sqrt{y-x}\sqrt{s-y}}$$

$$= -\frac{1}{\pi}\frac{d}{dx}\int_x^{z_0} ds\, G(s) \int_x^{s} dy \left[\left(\frac{s-x}{2}\right)^2 - \left(y - \frac{s+x}{2}\right)^2\right]^{-1/2}$$

$$= -\frac{1}{\pi}\frac{d}{dx}\int_x^{z_0} ds\, G(s) \int_{-\frac{s-x}{2}}^{\frac{s-x}{2}} du \left[\left(\frac{s-x}{2}\right)^2 - u^2\right]^{-1/2}$$

$$= -\frac{1}{\pi}\frac{d}{dx}\int_x^{z_0} ds\, G(s) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = -\frac{d}{dx}\int_x^{z_0} ds\, G(s) = G(x)$$

$$G(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^{z_0} \frac{dy}{\sqrt{y-x}}\, F(y)$$

$$= \frac{1}{\pi}\left(\lim_{x'\to x} \frac{F(x')}{\sqrt{x'-x}}\right) - \frac{1}{\pi}\int_x^{z_0} \frac{d}{dx}\left(\frac{1}{\sqrt{y-x}}\right) F(y)\, dy$$

$$= \frac{1}{\pi}\left(\lim_{x'\to x} \frac{F(x')}{\sqrt{x'-x}}\right) + \frac{1}{\pi}\int_x^{z_0} \frac{d}{dy}\left(\frac{1}{\sqrt{y-x}}\right) F(y)\, dy$$

$$G(x) = \frac{1}{\pi}\left(\lim_{x'\to z_0} \frac{F(x')}{\sqrt{x'-x}}\right) - \frac{1}{\pi}\int_x^{z_0} \frac{dy}{\sqrt{y-x}}\frac{dF}{dy}$$

can drop first term if F(y) → 0 at boundary

Eddington inversion as an Abel integral transform

ρ(r) and Φ(r) are monotonic, so we get ρ(Φ)

f(E) , ρ(Φ) = 0 for E, Φ > Φ(∞)

$$F(y)=\int_y^\infty \frac{dx}{\sqrt{x-y}}G(x)$$

$$G(x)=-\frac{1}{\pi}\int_x^\infty \frac{dy}{\sqrt{y-x}}\frac{dF}{dy}$$

$$y=E, \qquad x=\Phi$$

$$F(y)=f(E), \qquad G(\Phi)=\frac{1}{2\sqrt{2}\pi^2}\frac{d^2\rho}{d\Phi^2}$$

$$\frac{d^2\rho}{d\Phi^2}=-2\sqrt{2}\pi\int_\Phi^\infty \frac{dE}{\sqrt{E-\Phi}}\frac{df}{dE}$$

$$\rho=-2\sqrt{2}\pi\int_\Phi^\infty d\Phi'\int_{\Phi'}^\infty d\Phi''\int_{\Phi''}^\infty \frac{dE}{\sqrt{E-\Phi''}}\frac{df}{dE}$$

$$=-2\sqrt{2}\pi\int_\Phi^\infty dE\,\frac{df}{dE}\int_\Phi^E d\Phi'\int_{\Phi'}^E \frac{d\Phi''}{\sqrt{E-\Phi''}}$$

$$=-4\sqrt{2}\pi\int_\Phi^\infty dE\,\frac{df}{dE}\int_\Phi^E d\Phi'\sqrt{E-\Phi'}$$

$$=-\frac{8}{3}\sqrt{2}\pi\int_\Phi^\infty dE\,\frac{df}{dE}(E-\Phi)^{3/2}$$

$$\rho=4\sqrt{2}\pi\int_\Phi^\infty dE\, f\,\sqrt{E-\Phi}$$

(need F(E)→0 faster than $E^{-3/2}$ as E → ∞)

canonical transformations and symplectic Jacobians

$$\eta = \begin{pmatrix} x \\ p \end{pmatrix}, \quad \xi = \begin{pmatrix} X \\ P \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_{3\times 3} \\ -I_{3\times 3} & 0 \end{pmatrix}$$

$$\xi = \xi(\eta)$$

$$M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}, \quad M_{ij}^T = \frac{\partial \xi_j}{\partial \eta_i}$$

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$$\dot{\eta} = J\frac{\partial H}{\partial \eta}$$

$$\dot{\xi}_i = \sum_j \frac{\partial \xi_i}{\partial \eta_j}\dot{\eta}_j$$

$$\frac{\partial H}{\partial \eta_i} = \sum_j \frac{\partial \xi_j}{\partial \eta_i}\frac{\partial H}{\partial \xi_j}$$

$$\dot{\xi} = M\dot{\eta} = MJ\frac{\partial H}{\partial \eta} = MJM^T\frac{\partial H}{\partial \xi}$$

$$= J\frac{\partial H}{\partial \xi}$$

$$J = MJM^T$$

time translation as an infinitesimal canonical transformation

$$X_i(t) = x_i(t+dt) = x_i(t) + \dot{x}_i(t)dt = x_i + dt(\partial H/\partial p_i)$$

$$P_i(t) = p_i(t+dt) = p_i(t) + \dot{p}_i(t)dt = p_i - dt(\partial H/\partial x_i)$$

$$M = \begin{pmatrix} \delta_{ij} + dt\partial^2 H/\partial p_i \partial x_j & dt\partial^2 H/\partial p_i \partial p_j \\ -dt\partial^2 H/\partial x_i \partial x_j & \delta_{ij} - dt\partial^2 H/\partial x_i \partial p_j \end{pmatrix}$$

$$M^T = \begin{pmatrix} \delta_{ij} + dt\partial^2 H/\partial x_i \partial p_j & -dt\partial^2 H/\partial x_i \partial x_j \\ dt\partial^2 H/\partial p_i \partial p_j & \delta_{ij} - dt\partial^2 H/\partial p_i \partial x_j \end{pmatrix}$$

$$M^T J M = \begin{pmatrix} \delta_{ij} + dt\partial^2 H/\partial x_i \partial p_j & -dt\partial^2 H/\partial x_i \partial x_j \\ dt\partial^2 H/\partial p_i \partial p_j & \delta_{ij} - dt\partial^2 H/\partial p_i \partial x_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{ij} + dt\partial^2 H/\partial p_i \partial x_j & dt\partial^2 H/\partial p_i \partial p_j \\ -dt\partial^2 H/\partial x_i \partial x_j & \delta_{ij} - dt\partial^2 H/\partial x_i \partial p_j \end{pmatrix}$$

$$= \begin{pmatrix} \delta_{ij} + dt\partial^2 H/\partial x_i \partial p_j & -dt\partial^2 H/\partial x_i \partial x_j \\ dt\partial^2 H/\partial p_i \partial p_j & \delta_{ij} - dt\partial^2 H/\partial p_i \partial x_j \end{pmatrix} \begin{pmatrix} -dt\partial^2 H/\partial x_i \partial x_j & \delta_{ij} - dt\partial^2 H/\partial x_i \partial p_j \\ -\delta_{ij} - dt\partial^2 H/\partial p_i \partial x_j & -dt\partial^2 H/\partial p_i \partial p_j \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I_{3\times 3} \\ -I_{3\times 3} & 0 \end{pmatrix} + O(dt^2)$$

include baryons